Online Appendix to
'Trade and Interdependence
in a Spatially Complex World'

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M The impact of border costs in the large-space limit on the circle

This appendix derives the limit behavior (47) and (48) of the impact of border costs in the case of trade costs given by $\tilde{\tau}(d) = \left(1 + 4\alpha^2 R^2 \sin^2 \frac{d}{2R}\right)^{\frac{\varrho}{2}}$. The corresponding expression for $G_{c,n}$ is given in (44). $G_{c,n}$ is independent of the sign of n, and is a decreasing function of $|n|$. The quantity Z, used frequently in this appendix, is defined as $Z \equiv 1/\sqrt{1 + 4\alpha^2 R^2}$.

M.1 Nonnegativity of $(-1)^{\frac{n}{2}} \tilde{g}_{c,n}$

Let us show that $(-1)^{\frac{n}{2}} \tilde{g}_{c,n} \geq 0$. The Fourier coefficients $\tilde{g}_{c,n}$ are given by (45). All $\tilde{g}_{c,n}$ with odd n vanish. Also, $\tilde{g}_{c,0} \geq 0$ because $\tilde{g}_c(\theta)$ is nonnegative for any θ. This means that it is sufficient to show that

$$\sum_{m=0}^{\infty} \frac{LG_{c,2m+1}}{(2m+1)^2 - n^2} \tag{81}$$

is nonpositive for even nonzero n. For this purpose, we can use the identity

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2 - n^2} = \frac{1}{2n} \sum_{m=0}^{\infty} \left(\frac{1}{2m+1-n} - \frac{1}{2m+1+n} \right) = \frac{1}{2n} \sum_{m=-\frac{n}{2}}^{\frac{n}{2}-1} \frac{1}{2m+1} = 0$$

[58]To be used with the version of the main paper from the same date. This is not the final version of the online appendix. The online appendix will be expanded over time to include significantly more material.

to rewrite the term of interest (81) as

$$\sum_{m=0}^{\infty} \frac{LG_{c,2m+1} - LG_{c,n}}{(2m+1)^2 - n^2}.$$

If $2m+1 < n$, both the numerator and the denominator are negative. If $2m+1 > n$, they are both positive. This means that all contributions to the infinite sum are positive. We conclude that $(-1)^{\frac{n}{2}} \tilde{g}_{c,n} \geq 0$.

M.2 A lower bound on $-y_1^{(P)}\left(\frac{\pi}{2}\right)$

Plugging the expressions (41) for $y_{1,n}^{(P)}$ into the general formula for Fourier series expansion (37) gives

$$-\frac{y_1^{(P)}\left(\frac{\pi}{2}\right)}{y_0^{(P)}} = -\frac{1}{y_0^{(P)}} \sum_{n=-\infty}^{\infty} y_{1,n}^{(P)} e^{in\frac{\pi}{2}} = \frac{\sigma}{\sigma-1}\tilde{g}_{c,0} + \frac{2\sigma-1}{\sigma-1} \sum_{n \text{ even nonzero}} \frac{(-1)^{\frac{n}{2}} \tilde{g}_{c,n}}{1+(\sigma-1)\,LG_{c,n}}. \tag{82}$$

We know from Subsection 5.4 that $LG_{c,0} = 1/\sigma$ and from Subsection M.1 that $(-1)^{\frac{n}{2}} \tilde{g}_{c,n} \geq 0$. Equation (82) combined with $0 \leq LG_{c,n} \leq 1/\sigma$ and $(-1)^{\frac{n}{2}} \tilde{g}_{c,n} \geq 0$ implies

$$-\frac{y_1^{(P)}\left(\frac{\pi}{2}\right)}{y_0^{(P)}} \geq \frac{\sigma}{\sigma-1} \sum_{n \text{ even}} (-1)^{\frac{n}{2}} \tilde{g}_{c,n}.$$

Taking into account the symmetry between the two countries, the fact that $\int G_c(\theta)\, dL(\theta) = LG_{c,0} = 1/\sigma$, and the definition (24) of $\tilde{g}_c$, we see that

$$\sum_{n \text{ even}} (-1)^{\frac{n}{2}} \tilde{g}_{c,n} = \frac{1}{2\sigma}. \tag{83}$$

This leads to the following lower bound on $-y_1^{(P)}\left(\frac{\pi}{2}\right)$:

$$-\frac{y_1^{(P)}\left(\frac{\pi}{2}\right)}{y_0^{(P)}} \geq \frac{1}{2}\frac{1}{\sigma-1}. \tag{84}$$

M.3 An upper bound on $-y_1^{(P)}\left(\frac{\pi}{2}\right)$

Equation (82) may be used to derive also an upper bound on $-y_1^{(P)}\left(\frac{\pi}{2}\right)$. Using $LG_{c,n} \geq 0$ and $(-1)^{\frac{n}{2}}\,\tilde{g}_{c,n} \geq 0$ leads to

$$-\frac{y_1^{(P)}\left(\frac{\pi}{2}\right)}{y_0^{(P)}} \leq -\tilde{g}_{c,0} + \frac{2\sigma-1}{\sigma-1}\sum_{n \text{ even}}(-1)^{\frac{n}{2}}\,\tilde{g}_{c,n}.$$

Omitting the first term on the right-hand side and simplifying the second term using (83) gives

$$-\frac{y_1^{(P)}\left(\frac{\pi}{2}\right)}{y_0^{(P)}} \leq \frac{2\sigma-1}{2\sigma}\frac{1}{\sigma-1}. \tag{85}$$

M.4 A lower bound on $-\lim_{R\to\infty} y_1^{(P)}(0)$ for $\delta < \frac{1}{2}$

This subsection contains the derivation of a lower bound on $-\lim_{R\to\infty} y_1^{(P)}(0)$. To simplify notation, the limit symbol $\lim_{R\to\infty}$ will be omitted, but its presence is implicitly understood.

M.4.1 The asymptotic form of $G_{c,n}$ for $\delta < \frac{1}{2}$

For arbitrary R, the expression (44) for $G_{c,n}$ is

$$G_{c,n} = \frac{1}{\sigma L}\frac{(-1)^n}{(1-\delta)_n}\frac{P_{\delta-1}^{n}\left(\frac{1+Z^2}{2Z}\right)}{P_{\delta-1}\left(\frac{1+Z^2}{2Z}\right)}.$$

In the large R limit the expression simplifies to

$$G_{c,n} = \frac{1}{\sigma L}\frac{(\delta)_n}{(1-\delta)_n}. \tag{86}$$

This asymptotic form can be verified using the definition of the Pochhammer symbol

$$(\delta)_n \equiv \frac{\Gamma(\delta+n)}{\Gamma(\delta)} = (-1)^n\frac{\Gamma(1-\delta)}{\Gamma(1-\delta-n)},$$

and the equation 8.766(1) on p. 971 of Gradshteyn and Ryzhik (2007), which states that for $|z| \gg 1$,

$$P_{\nu}^{\mu}(z) = \left\{ \frac{2^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(\nu-\mu+1)} z^{\nu} + \frac{\Gamma\left(-\nu-\frac{1}{2}\right)}{2^{\nu+1}\sqrt{\pi}\Gamma(-\nu-\mu)} z^{-\nu-1} \right\} \left(1 + O\left(\frac{1}{z^2}\right)\right).$$

M.4.2 The Fourier series expansion of $y_1^{(P)}(0)$

Using the expressions (46) for $y_{1,n}^{(P)}$ in the general formula for Fourier series expansion (38) gives

$$\begin{aligned} -\frac{y_1^{(P)}(0)}{y_0^{(P)}} &= \frac{1}{2}\frac{1}{\sigma-1} - \frac{4}{\pi^2}\frac{\sigma}{\sigma-1}\sum_{m=0}^{\infty}\frac{LG_{c,2m+1}}{(2m+1)^2} \\ &\quad -\frac{4}{\pi^2}\frac{2\sigma-1}{\sigma-1}\sum_{n \text{ even nonzero}}\frac{(-1)^{\frac{n}{2}}}{1+(\sigma-1)LG_{c,n}}\sum_{m=0}^{\infty}\frac{LG_{c,2m+1}}{(2m+1)^2-n^2}. \end{aligned}$$

This relation may be rewritten as

$$\begin{aligned} -\frac{y_1^{(P)}(0)}{y_0^{(P)}} &= -\left(\frac{1}{2\sigma} - \frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}LG_{c,2m+1}\right) \\ &\quad +\frac{2\sigma-1}{\sigma-1}\left(\frac{1}{2\sigma} - \frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}LG_{c,2m+1}\right) \\ &\quad -\frac{4}{\pi^2}\frac{2\sigma-1}{\sigma-1}\sum_{n \text{ even nonzero}}(-1)^{\frac{n}{2}}\sum_{m=0}^{\infty}\frac{LG_{c,2m+1}}{(2m+1)^2-n^2} \\ &\quad +\frac{4}{\pi^2}\frac{2\sigma-1}{\sigma-1}\sum_{n \text{ even nonzero}}\frac{(-1)^{\frac{n}{2}}(\sigma-1)LG_{c,n}}{1+(\sigma-1)LG_{c,n}}\sum_{m=0}^{\infty}\frac{LG_{c,2m+1}}{(2m+1)^2-n^2}. \end{aligned}$$

The first term on the right-hand side is just $-\tilde{g}_{c,0}$ due to (45). The terms on the second and third line add up to $\frac{2\sigma-1}{\sigma-1}\tilde{g}_c(0)$, as implied by the Fourier series expansion formula (37) with the Fourier coefficients (45):

$$\tilde{g}_c(0) = \frac{1}{2\sigma} - \frac{4}{\pi^2}\sum_{n \text{ even}}(-1)^{\frac{n}{2}}\sum_{m=0}^{\infty}\frac{LG_{c,2m+1}}{(2m+1)^2-n^2}. \tag{87}$$

The Fourier series expansion of $y_1^{(P)}(0)$ can now be written as

$$-\frac{y_1^{(P)}(0)}{y_0^{(P)}} = -\tilde{g}_{c,0} + \frac{2\sigma-1}{\sigma-1}\tilde{g}_c(0) + \frac{4}{\pi^2}\frac{2\sigma-1}{\sigma-1}\sum_{\substack{n \text{ even} \\ \text{nonzero}}} \frac{(-1)^{\frac{n}{2}}(\sigma-1)LG_{c,n}}{1+(\sigma-1)LG_{c,n}} \sum_{m=0}^{\infty} \frac{LG_{c,2m+1}}{(2m+1)^2 - n^2}. \tag{88}$$

The rest of this subsection analyzes the properties of the terms on the right-hand side in the large-space limit. The analysis then leads to an asymptotic lower bound on $-y_1^{(P)}(0)$.

M.4.3 Evaluating $\lim_{R\to\infty} \tilde{g}_c(0)$

The expression (87) for $\tilde{g}_c(0)$ with the asymptotic form (86) of $G_{c,n}$ is

$$\sigma\tilde{g}_c(0) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n \text{ even}} (-1)^{\frac{n}{2}} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2 - n^2} \frac{(\delta)_{2m+1}}{(1-\delta)_{2m+1}}.$$

The identity

$$\begin{aligned} &\sum_{n \text{ even}}^{\infty} \frac{4(-1)^{\frac{n}{2}}}{\pi^2} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2 - n^2} \frac{(\delta)_{2m+1}}{(1-\delta)_{2m+1}} \\ = &\ \frac{1}{2} - 2\frac{\Gamma(1-\delta)}{\sqrt{\pi}\Gamma\left(\frac{1}{2}-\delta\right)} F\left(\frac{1}{2}, 1-\delta; \frac{3}{2}; -1\right) \end{aligned}$$

then leads to the following compact result for $\tilde{g}_c(0)$:

$$\sigma\tilde{g}_c(0) = 2\frac{\Gamma(1-\delta)}{\sqrt{\pi}\Gamma\left(\frac{1}{2}-\delta\right)} F\left(\frac{1}{2}, 1-\delta; \frac{3}{2}; -1\right). \tag{89}$$

M.4.4 An alternative way of evaluating of $\lim_{R\to\infty} \tilde{g}_c(0)$

An alternative way of deriving (89) is to work directly with the definition (24) of $\tilde{g}_c(\theta)$, which implies

$$\tilde{g}_c(0) = 2\rho_L \int_{\frac{\pi}{2}}^{\pi} G_c(\theta)\, d\theta.$$

Substituting the expression (42) for $G_c(\theta)$, we get

$$\tilde{g}_c(0) = \frac{1}{\sigma}\frac{\int_{\frac{\pi}{2}}^{\pi} T(\theta)\, d\theta}{\int_0^{\pi} T(\theta)\, d\theta}.$$

For the functional form (43) of $T(\theta)$ this is

$$\tilde{g}_c(0) = \frac{1}{\sigma}\frac{\int_{\frac{\pi}{2}}^{\pi}\left(Z^2\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\right)^{-\delta}d\theta}{\int_0^{\pi}\left(Z^2\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\right)^{-\delta}d\theta}.$$

The large-space limit $R\to\infty$ corresponds to $Z\to 0_+$, and in this limit

$$\tilde{g}_c(0) = \frac{1}{\sigma}\frac{\int_{\frac{\pi}{2}}^{\pi}\sin^{-2\delta}\frac{\theta}{2}d\theta}{\int_0^{\pi}\sin^{-2\delta}\frac{\theta}{2}d\theta} = \frac{1}{\sigma}\frac{\int_{\frac{\pi}{2}}^{\pi}(1-\cos\theta)^{-\delta}d\theta}{\int_0^{\pi}(1-\cos\theta)^{-\delta}d\theta}.$$

To find an explicit expression for the integrals, we can use the substitution $t\equiv\frac{1+\cos\theta}{2}$, $d\theta = -\frac{1}{2}t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt$,

$$\tilde{g}_c(0) = \frac{1}{\sigma}\frac{\int_0^{\frac{1}{2}}(1-t)^{-\delta-\frac{1}{2}}t^{-\frac{1}{2}}dt}{\int_0^1(1-t)^{-\delta-\frac{1}{2}}t^{-\frac{1}{2}}dt} = \frac{1}{\sigma}\frac{\mathrm{B}_{\frac{1}{2}}\left(\frac{1}{2},\frac{1}{2}-\delta\right)}{\mathrm{B}_1\left(\frac{1}{2},\frac{1}{2}-\delta\right)}. \tag{90}$$

The second equality[59] here follows from the definition of the incomplete beta function,[60]

$$\mathrm{B}_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1}dt.$$

This special function should not be confused with the primary impact function (19). The result (90) for $\tilde{g}_c(0)$ matches the expression (89) derived by summing up the infinite series.

M.4.5 Evaluating $\lim_{R\to\infty}\tilde{g}_{c,0}$

Now let us look at $\tilde{g}_{c,0}$. The expression for $\tilde{g}_{c,0}$ in (45) with the asymptotic form (86) of $G_{c,n}$ becomes

$$\sigma\tilde{g}_{c,0} = \frac{1}{2} - \frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\frac{(\delta)_{2m+1}}{(1-\delta)_{2m+1}}.$$

[59]Note that $B_1\left(\frac{1}{2},\frac{1}{2}-\delta\right)$ may be written in terms of the (complete) beta function as $B\left(\frac{1}{2},\frac{1}{2}-\delta\right)$ or in terms of the gamma function as $\sqrt{\pi}\Gamma\left(\frac{1}{2}-\delta\right)/\Gamma(1-\delta)$. $B_{\frac{1}{2}}\left(\frac{1}{2},\frac{1}{2}-\delta\right)$ can be expressed in terms of the Gauss hypergeometric function as $\sqrt{2}F\left(\frac{1}{2},\frac{1}{2}+\delta;\frac{3}{2};\frac{1}{2}\right)$ or as $2F\left(\frac{1}{2},1-\delta;\frac{3}{2};-1\right)$.

[60]See, for example, equation eq. 8.391 on p. 910 of Gradshteyn and Ryzhik (2007).

The sum can be expressed in terms of the generalized hypergeometric function ${}_5F_4$,

$$\sigma\tilde{g}_{c,0} = \frac{1}{2} + \frac{4}{\pi^2}\frac{\delta}{1-\delta}\ {}_5F_4\left(\frac{1}{2},\frac{1}{2},1,\frac{1}{2}+\frac{\delta}{2},1+\frac{\delta}{2};\frac{3}{2},\frac{3}{2},1-\frac{\delta}{2},\frac{3}{2}-\frac{\delta}{2};1\right).$$

M.4.6 Positivity of the last term

Consider now the last term in (88):

$$\frac{4}{\pi^2}\frac{2\sigma-1}{\sigma-1}\sum_{n\text{ even nonzero}}\frac{(-1)^{\frac{n}{2}}(\sigma-1)\,LG_{c,n}}{1+(\sigma-1)\,LG_{c,n}}\sum_{m=0}^{\infty}\frac{LG_{c,2m+1}}{(2m+1)^2-n^2}.$$

With $G_{c,n}$ given by (86), the inner sum can be evaluated explicitly. For n even and nonzero,

$$\begin{aligned}
&\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2-n^2}\frac{(\delta)_{2m+1}}{(1-\delta)_{2m+1}}\\
=\ &-\frac{\delta}{2(1-\delta)}\frac{1}{n(n^2-1)}\\
&\left((n-1)\ {}_4F_3\left(1,\frac{1+n}{2},\frac{1+\delta}{2},\frac{1+\delta}{2};\frac{3+n}{2},1-\frac{\delta}{2},\frac{3}{2}-\frac{\delta}{2};1\right)\right.\\
&\left.+(n+1)\ {}_4F_3\left(1,\frac{1-n}{2},\frac{1+\delta}{2},\frac{1+\delta}{2};\frac{3-n}{2},1-\frac{\delta}{2},\frac{3}{2}-\frac{\delta}{2};1\right)\right).
\end{aligned}$$

Let us now restrict attention to $|n|$ even and nonzero. The expression $\sum_{m=0}^{\infty}\frac{LG_{c,2m+1}}{(2m+1)^2-n^2}$ is negative, and its absolute value is a decreasing function of $|n|$. Note also that $\frac{(\sigma-1)LG_{c,n}}{1+(\sigma-1)LG_{c,n}}$ is a positive decreasing function of $|n|$. The factor $(-1)^{\frac{n}{2}}$ is just an alternating sign, which is negative for the lowest terms, i.e. for $|n|=2$. These facts imply that the last term in (88) is positive. Omitting this term in (88) leads to a lower bound on $-y_1^{(P)}(0)$.

$$-\frac{y_1^{(P)}(0)}{y_0^{(P)}} \geq -\tilde{g}_{c,0} + \frac{2\sigma-1}{\sigma-1}\tilde{g}_c(0).$$

Noting that $\frac{2\sigma-1}{\sigma-1} \geq 2$ and $\tilde{g}_c(0) \geq 0$, this implies the weaker bound

$$-\frac{y_1^{(P)}(0)}{y_0^{(P)}} \geq 2\tilde{g}_c(0) - \tilde{g}_{c,0}.$$

M.4.7 The resulting lower bound on $\lim_{R\to\infty} y_1^{(P)}(0)$

Substituting the expressions for $\tilde{g}_{c,0}$ and $\tilde{g}_c(0)$ into the last inequality, we get

$$\begin{aligned} -\frac{\sigma y_1^{(P)}(0)}{y_0^{(P)}} &= \frac{4\Gamma(1-\delta)}{\sqrt{\pi}\Gamma\left(\frac{1}{2}-\delta\right)} F\left(\frac{1}{2}, 1-\delta; \frac{3}{2}; -1\right) \\ &\quad -\frac{1}{2} - \frac{4}{\pi^2}\frac{\delta}{1-\delta}\, {}_5F_4\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}+\frac{\delta}{2}, 1+\frac{\delta}{2}; \frac{3}{2}, \frac{3}{2}, 1-\frac{\delta}{2}, \frac{3}{2}-\frac{\delta}{2}; 1\right). \end{aligned}$$

The function on the right-hand side is concave, takes value of $\frac{1}{2}$ at $\delta = 0$, and vanishes at $\delta = \frac{1}{2}$. It is therefore never smaller than $\frac{1}{2} - \delta$. This leads to the result

$$-\frac{y_1^{(P)}(0)}{y_0^{(P)}} \geq \frac{\frac{1}{2}-\delta}{\sigma}. \qquad (91)$$

M.5 An alternative derivation of the bounds (84) and (85) in the large-space limit for $\delta > \frac{1}{2}$

M.5.1 The large space limit of $G_{c,n}$ for $\delta > \frac{1}{2}$

Consider again the expression (44) for $G_{c,n}$:

$$G_{c,n} = \frac{1}{\sigma L}\frac{(-1)^n}{(1-\delta)_n}\frac{P_{\delta-1}^n\left(\frac{1+Z^2}{2Z}\right)}{P_{\delta-1}\left(\frac{1+Z^2}{2Z}\right)},$$

with the goal of understanding the $R \to \infty$ (i.e. $Z \to 0_+$) limit when $\delta > \frac{1}{2}$. The definition[61] of $P_\nu^\mu(z)$ is

$$P_\nu^\mu(z) = \left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} {}_2\tilde{F}_1\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right),$$

where ${}_2\tilde{F}_1$ is the regularized hypergeometric function. In order to apply this relation directly to the expression for $G_{c,n}$, we need to choose

$$z = \frac{1+Z^2}{2Z}, \quad \frac{1-z}{2} = -\frac{(1-Z)^2}{4Z}, \quad \frac{z+1}{z-1} = \left(\frac{1+Z}{1-Z}\right)^2.$$

[61] See http://functions.wolfram.com/07.09.02.0001.01

The Fourier coefficients $G_{c,n}$ become

$$G_{c,n} = \frac{1}{\sigma L}\frac{(-1)^n}{(1-\delta)_n}\left(\frac{1+Z}{1-Z}\right)^n \frac{{}_2\tilde{F}_1\left(1-\delta,\delta;1-n;-\frac{(1-Z)^2}{4Z}\right)}{{}_2\tilde{F}_1\left(1-\delta,\delta;1;-\frac{(1-Z)^2}{4Z}\right)}.$$

Since $G_{c,n} = G_{c,-n}$, we know that the expression on the right-hand side does not depend on the sign of n. It is therefore sufficient to focus on non-positive n. In this case one can use the ordinary hypergeometric function ${}_2F_1$ instead of the regularized one. These functions are related[62] by ${}_2\tilde{F}_1(a,b,c,x) \equiv {}_2F_1(a,b,c,x)/\Gamma(c)$, where $\Gamma(x)$ is the gamma function.

$$G_{c,n} = \frac{1}{\sigma L}\frac{(-1)^n}{(1-\delta)_n}\left(\frac{1+Z}{1-Z}\right)^n \frac{1}{\Gamma(1-n)}\frac{{}_2F_1\left(1-\delta,\delta;1-n;-\frac{(1-Z)^2}{4Z}\right)}{{}_2F_1\left(1-\delta,\delta;1;-\frac{(1-Z)^2}{4Z}\right)}.$$

For convenience, define the rescaled Fourier mode number[63] ω as

$$\omega \equiv -Zn.$$

In terms of ω, the Fourier coefficients of G_c are

$$G_{c,\frac{\omega}{Z}} = \frac{1}{\sigma L}\left(\frac{1+Z}{1-Z}\right)^{-\frac{\omega}{Z}} \times \frac{(-1)^{\frac{\omega}{Z}}}{(1-\delta)_{-\frac{\omega}{Z}}}\frac{Z^{\delta-1}}{\Gamma\left(1+\frac{\omega}{Z}\right)} \times \frac{{}_2F_1\left(1-\delta,\delta;1+\frac{\omega}{Z};-\frac{(1-Z)^2}{4Z}\right)}{Z^{\delta-1}\,{}_2F_1\left(1-\delta,\delta;1;-\frac{(1-Z)^2}{4Z}\right)}. \quad (92)$$

The right-hand side has been split into three factors. Let us look at the $Z \to 0_+$ limit (holding ω fixed) of each of them separately.

The first factor in (92) has a very simple limit. The definition of the exponential function implies

$$\lim_{Z\to 0}\frac{1}{\sigma L}\left(\frac{1+Z}{1-Z}\right)^{-\frac{\omega}{Z}} = \frac{1}{\sigma L}e^{-2\omega}.$$

To evaluate the limit of the second factor in (92), recall the definition of the Pochham-

[62]See http://functions.wolfram.com/07.24.26.0003.01
[63]Not to be confused with the notation for different varieties of goods.

mer symbol

$$(1-\delta)_{-\frac{\omega}{Z}} \equiv \frac{\Gamma\left(1-\delta-\frac{\omega}{Z}\right)}{\Gamma\left(1-\delta\right)}.$$

With the help of the gamma function identity[64] $\Gamma(1-x)\Gamma(x)\sin\pi x=\pi$, this is

$$(1-\delta)_{-\frac{\omega}{Z}} = \frac{1}{\Gamma(1-\delta)}\frac{1}{\Gamma\left(\delta+\frac{\omega}{Z}\right)}\frac{\pi}{\sin\pi\left(\delta+\frac{\omega}{Z}\right)} = (-1)^{-\frac{\omega}{Z}}\frac{1}{\Gamma(1-\delta)}\frac{1}{\Gamma\left(\delta+\frac{\omega}{Z}\right)}\frac{\pi}{\sin\pi\delta},$$

where the second equality follows from the periodicity properties of the sine function with $\frac{\omega}{Z}\in\mathbb{Z}$. The desired limit is

$$\lim_{Z\to0_+}\frac{(-1)^{-\frac{\omega}{Z}}}{(1-\delta)_{-\frac{\omega}{Z}}}\frac{Z^{\delta-1}}{\Gamma\left(1+\frac{\omega}{Z}\right)} = \Gamma(1-\delta)\frac{\sin\pi\delta}{\pi}\lim_{Z\to0_+}\frac{Z^{\delta-1}\Gamma\left(\delta+\frac{\omega}{Z}\right)}{\Gamma\left(1+\frac{\omega}{Z}\right)} = \Gamma(1-\delta)\frac{\sin\pi\delta}{\pi}\omega^{\delta-1}.$$

The second equality may be verified using Stirling's formula.

To find the limit of the third factor in (92), we need two identities. The first one[65] reads

$$\lim_{c\to\infty}\ {}_2F_1\left(a,a-b+1;c;1-\frac{c}{z}\right)=z^aU(a,b,z).$$

The correct version of the second one[66] is

$$U(a,2a,z)=\frac{1}{\sqrt{\pi}}z^{\frac{1}{2}-a}e^{\frac{z}{2}}K_{a-\frac{1}{2}}\left(\frac{z}{2}\right).$$

The function U is the confluent hypergeometric function of the second kind, and K is the modified Bessel function of the second kind. Combining the two identities, and using $K_\nu(x)=K_{-v}(x)$ yields

$$\lim_{c\to\infty}\ {}_2F_1\left(a,1-a;c;1-\frac{c}{z}\right)=\frac{1}{\sqrt{\pi}}z^{\frac{1}{2}}e^{\frac{z}{2}}K_{\frac{1}{2}-a}\left(\frac{z}{2}\right).$$

The choice

$$a=1-\delta,\ c=1+\frac{\omega}{Z},\ z=4\omega$$

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[64] See, for example, equation 8.334(3) on p. 896 of Gradshteyn and Ryzhik (2007).

[65] Available at http://functions.wolfram.com/07.33.09.0001.01

[66] Available at http://functions.wolfram.com/07.33.03.0007.01
Note that the graphical version of the formula is wrong on the website, but its Mathematica version is correct.

in this formula leads to

$$\begin{aligned}\lim_{Z\to 0_+} {}_2F_1\left(1-\delta,\delta;1+\frac{\omega}{Z};-\frac{(1-Z)^2}{4Z}\right) &= \lim_{Z\to 0_+} {}_2F_1\left(1-\delta,\delta;1+\frac{\omega}{Z};\frac{4\omega-1}{4\omega}-\frac{1}{4Z}\right)\\ &= \frac{2}{\sqrt{\pi}}e^{2\omega}\omega^{\frac{1}{2}}K_{\delta-\frac{1}{2}}(2\omega).\end{aligned}$$

For the denominator of the third factor in (92), asymptotic properties of hypergeometric functions imply

$$\lim_{Z\to 0_+} Z^{\delta-1}\ {}_2F_1\left(1-\delta,\delta;1;-\frac{(1-Z)^2}{4Z}\right) = 4^{1-\delta}\frac{\Gamma(2\delta-1)}{\Gamma^2(\delta)}.$$

We have

$$\lim_{Z\to 0_+} \frac{{}_2F_1\left(1-\delta,\delta;1+\frac{\omega}{Z};-\frac{(1-Z)^2}{4Z}\right)}{Z^{\delta-1}\ {}_2F_1\left(1-\delta,\delta;1;-\frac{(1-Z)^2}{4Z}\right)} = \frac{\frac{2}{\sqrt{\pi}}e^{2\omega}\omega^{\frac{1}{2}}K_{\delta-\frac{1}{2}}(2\omega)}{4^{1-\delta}\frac{\Gamma(2\delta-1)}{\Gamma^2(\delta)}}$$

Applying these three results to (92) gives

$$G_{c,\frac{\omega}{Z}} = \frac{1}{\sigma L}\frac{2^{2\delta-1}\Gamma(\delta)}{\sqrt{\pi}\Gamma(2\delta-1)}\Gamma(1-\delta)\Gamma(\delta)\frac{\sin\pi\delta}{\pi}\omega^{\delta-\frac{1}{2}}K_{\delta-\frac{1}{2}}(2\omega).$$

Using the gamma function identities[67]

$$\Gamma(1-\delta)\Gamma(\delta)\frac{\sin\pi\delta}{\pi} = 1$$

and

$$\frac{2^{2\delta-1}\Gamma(\delta)}{\sqrt{\pi}\Gamma(2\delta-1)} = \frac{2}{\Gamma\left(\delta-\frac{1}{2}\right)},$$

the Fourier coefficients simplify to

$$G_{c,\frac{\omega}{Z}} = \frac{1}{\sigma L}\frac{2}{\Gamma\left(\delta-\frac{1}{2}\right)}\omega^{\delta-\frac{1}{2}}K_{\delta-\frac{1}{2}}(2\omega).$$

[67]See, for example, equations 8.334(3) and 8.335(1) on p. 896 of Gradshteyn and Ryzhik (2007).

We conclude that for arbitrary ω, in the large-space limit the Fourier coefficients become

$$G_{c,\frac{\omega}{Z}} = \frac{1}{\sigma L}\frac{2}{\Gamma\left(\delta-\frac{1}{2}\right)}\left|\omega\right|^{\delta-\frac{1}{2}} K_{\delta-\frac{1}{2}}\left(2\left|\omega\right|\right). \tag{93}$$

Notice that the properties of the modified Bessel function of the second kind imply that the expression on the right-hand side approaches $1/\left(\sigma L\right)$ when $\omega \to 0$, as expected from the general relation $\sigma L G_{c,0} = 1$.

M.5.2 The limit of $y^{(P)}\left(\frac{\pi}{2}\right)$ for $\delta > \frac{1}{2}$

The general formula for Fourier series expansion (38) together with the expressions (46) for $y_{1,n}^{(P)}$ implies

$$\begin{aligned} -\left(\sigma-1\right)\frac{y_1^{(P)}\left(\frac{\pi}{2}\right)}{y_0^{(P)}} &= \frac{1}{2}-\frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{\sigma L G_{c,2m+1}}{\left(2m+1\right)^2} \\ &\quad -\sum_{n \text{ even nonzero}}\frac{2\sigma-1}{1+\left(\sigma-1\right)LG_{c,n}}\frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{LG_{c,2m+1}}{\left(2m+1\right)^2-n^2}. \end{aligned}$$

The $R \to \infty$ limit, or equivalently the $Z \to 0_+$ limit, of the first line on the right-hand side vanishes.[68] We have

$$-\left(\sigma-1\right)\lim_{R\to\infty}\frac{y_1^{(P)}\left(\frac{\pi}{2}\right)}{y_0^{(P)}} = -\lim_{R\to\infty}\sum_{n \text{ even nonzero}}\frac{2\sigma-1}{1+\left(\sigma-1\right)LG_{c,n}}\frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{LG_{c,2m+1}}{\left(2m+1\right)^2-n^2}.$$

In the $Z \to 0_+$ limit the sums can be faithfully approximated by integrals. Symbolically,

$$\sum_{m=0}^{\infty} f\left(2m+1\right) \to \frac{1}{4Z}\int f\left(\omega'\right)d\omega', \text{ for an even function } f,$$

$$\sum_{n \text{ even nonzero}} \to \frac{1}{2Z}\int d\omega.$$

[68] As $Z \to 0_+$, the coefficients $G_{c,2m+1}$ approach $1/\left(\sigma L\right)$, and the convergence is uniform in the appropriate sense. Also, $\sum_{m=0}^{\infty}\left(2m+1\right)^{-2} = \pi^2/8$.

More precisely, the integral over ω' should be taken in the sense of the Cauchy principal value (denoted p.v.). This gives

$$-\left(\sigma-1\right)\lim_{R\to\infty}\frac{y_1^{(P)}\left(\frac{\pi}{2}\right)}{y_0^{(P)}}=-\frac{1}{2\pi^2}\lim_{R\to\infty}\int\frac{2\sigma-1}{1+\left(\sigma-1\right)LG_{c,\frac{\omega}{Z}}}\text{p.v.}\int\frac{LG_{c,\frac{\omega'}{Z}}}{\omega'^2-\omega^2}d\omega' d\omega.$$

Using the algebraic relation

$$\frac{1}{\omega'^2-\omega^2}=-\frac{1}{2\omega}\frac{1}{\omega-\omega'}-\frac{1}{2\omega}\frac{1}{\omega+\omega'}$$

and the explicit expression (93) for $G_{c,\frac{\omega'}{Z}}$, we obtain

$$\begin{aligned}&-\left(\sigma-1\right)\lim_{R\to\infty}\frac{y_1^{(P)}\left(\frac{\pi}{2}\right)}{y_0^{(P)}}\\ =\;&\frac{1}{\sigma L}\frac{1}{\pi^2\Gamma\left(\delta-\frac{1}{2}\right)}\int\frac{2\sigma-1}{1+\frac{\sigma-1}{\sigma}\frac{2}{\Gamma\left(\delta-\frac{1}{2}\right)}\left|\omega\right|^{\delta-\frac{1}{2}}K_{\delta-\frac{1}{2}}\left(2\left|\omega\right|\right)}\frac{1}{\omega}\text{p.v.}\int\frac{\left|\omega'\right|^{\delta-\frac{1}{2}}K_{\delta-\frac{1}{2}}\left(2\left|\omega'\right|\right)}{\omega-\omega'}d\omega' d\omega.\end{aligned}$$

The integral over ω' can be evaluated explicitly,

$$\begin{aligned}&\text{p.v.}\int\tfrac{1}{\omega-\omega'}\left|\omega'\right|^{\delta-\frac{1}{2}}K_{\delta-\frac{1}{2}}\left(2\left|\omega'\right|\right)d\omega'\\ &=\tfrac{1}{4\pi}\tfrac{\omega}{|\omega|}\left|\omega\right|^{\delta-\frac{1}{2}}G_{4,2}^{4,2}\left(\omega^2\left|\begin{array}{c}\frac{1}{4}\left(1-2\delta\right),\frac{1}{4}\left(3-2\delta\right)\\ \frac{1}{4}\left(1-2\delta\right),\frac{1}{4}\left(1-2\delta\right),\frac{1}{4}\left(3-2\delta\right),-\frac{1}{4}\left(1-2\delta\right)\end{array}\right.\right)\\ &+\tfrac{\pi}{2}\tfrac{\omega}{|\omega|}\left|\omega\right|^{\delta-\frac{1}{2}}G_{6,4}^{4,2}\left(\omega^2\left|\begin{array}{cc}\frac{1}{4}\left(1-2\delta\right),\frac{1}{4}\left(3-2\delta\right), & \frac{1}{2}\left(1-\delta\right),-\frac{1}{2}\delta\\ \frac{1}{4}\left(1-2\delta\right),\frac{1}{4}\left(1-2\delta\right),\frac{1}{4}\left(3-2\delta\right),-\frac{1}{4}\left(1-2\delta\right), & \frac{1}{2}\left(1-\delta\right),-\frac{1}{2}\delta\end{array}\right.\right),\end{aligned}$$

where $G_{p,q}^{m,n}$ is the Meijer G-function. The outer integral over ω most likely does not lead to a closed form expression, but it can easily be evaluated numerically.

Since $\frac{1}{\omega}\text{p.v.}\int\frac{|\omega'|^{\delta-\frac{1}{2}}K_{\delta-\frac{1}{2}}(2|\omega'|)}{\omega-\omega'}d\omega'$ is positive, and $\frac{2}{\Gamma\left(\delta-\frac{1}{2}\right)}\left|\omega\right|^{\delta-\frac{1}{2}}K_{\delta-\frac{1}{2}}\left(2\left|\omega\right|\right)\in\left(0,1\right)$, one can immediately write the following bound on $y_1^{(P)}\left(\frac{\pi}{2}\right)$ in the large-space limit.

$$\begin{aligned} & \frac{1}{\sigma L}\frac{1}{\pi^2\Gamma\left(\delta-\frac{1}{2}\right)}\sigma\int\frac{1}{\omega}\text{p.v.}\int\frac{\left|\omega'\right|^{\delta-\frac{1}{2}}K_{\delta-\frac{1}{2}}\left(2\left|\omega'\right|\right)}{\omega-\omega'}d\omega' d\omega \leq \\ & -\left(\sigma-1\right)\lim_{R\to\infty}\frac{y_1^{(P)}\left(\frac{\pi}{2}\right)}{y_0^{(P)}} \\ \leq\ & \frac{1}{\sigma L}\frac{1}{\pi^2\Gamma\left(\delta-\frac{1}{2}\right)}\left(2\sigma-1\right)\int\frac{1}{\omega}\text{p.v.}\int\frac{\left|\omega'\right|^{\delta-\frac{1}{2}}K_{\delta-\frac{1}{2}}\left(2\left|\omega'\right|\right)}{\omega-\omega'}d\omega' d\omega. \end{aligned}$$

These integrals can be evaluated using the formula,[69]

$$\int_{-\infty}^{\infty}\frac{1}{\omega}\text{p.v.}\int_{-\infty}^{\infty}\frac{f\left(\omega'\right)}{\omega-\omega'}d\omega' d\omega=\pi^2 f\left(0\right),$$

which leads to

$$\frac{1}{2}\leq-\left(\sigma-1\right)\lim_{R\to\infty}\frac{y_1^{(P)}\left(\frac{\pi}{2}\right)}{y_0^{(P)}}\leq\frac{2\sigma-1}{2\sigma}.$$

This is, of course, consistent with the bounds (84) and (85) derived previously in a more general context.

M.6 Evaluating $\lim_{R\to\infty} y^{(P)}\left(0\right)$ for $\delta>\frac{1}{2}$

The logic used above to evaluate the large-space limit of $y_1^{(P)}\left(\frac{\pi}{2}\right)$ will be useful for finding the same limit of $y_1^{(P)}\left(0\right)$. The Fourier series expansion of $y_1^{(P)}\left(0\right)$ is

$$-\left(\sigma-1\right)\frac{y_1^{(P)}\left(0\right)}{y_0^{(P)}}=\frac{1}{2}-\sum_{n\text{ even}}\frac{2\sigma-1}{1+\left(\sigma-1\right)LG_{c,n}}\frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{\left(-1\right)^{\frac{n}{2}}LG_{c,2m+1}}{\left(2m+1\right)^2-n^2}.$$

Again, in the $R\to\infty$ limit, the $n=0$ term in the sum cancels against the $1/2$.

$$-\left(\sigma-1\right)\lim_{R\to\infty}\frac{y_1^{(P)}\left(0\right)}{y_0^{(P)}}=-\lim_{R\to\infty}\sum_{n\text{ even nonzero}}\frac{2\sigma-1}{1+\left(\sigma-1\right)LG_{c,n}}\frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{\left(-1\right)^{\frac{n}{2}}LG_{c,2m+1}}{\left(2m+1\right)^2-n^2}.$$

[69]See, for example, equation (8.4.15) of Kanwal (1997).

Splitting the sum into positive and negative contributions,

$$\begin{aligned}
-(\sigma-1)\lim_{R\to\infty}\frac{y_1^{(P)}(0)}{y_0^{(P)}} &= \lim_{R\to\infty}\sum_{\substack{n\text{ even nonzero}\\ n/2\text{ odd}}}\frac{2\sigma-1}{1+(\sigma-1)LG_{c,n}}\frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{LG_{c,2m+1}}{(2m+1)^2-n^2} \\
&\quad -\lim_{R\to\infty}\sum_{\substack{n\text{ even nonzero}\\ n/2\text{ even}}}^{\infty}\frac{2\sigma-1}{1+(\sigma-1)LG_{c,n}}\frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{LG_{c,2m+1}}{(2m+1)^2-n^2}.
\end{aligned}$$

The first sum on the right-hand side becomes

$$\frac{1}{2}\frac{1}{2\pi^2}\lim_{R\to\infty}\int\frac{2\sigma-1}{1+(\sigma-1)LG_{c,\frac{\omega}{Z}}}\text{p.v.}\int\frac{LG_{c,\frac{\omega'}{Z}}}{\omega'^2-\omega^2}d\omega'd\omega,$$

while the second sum is

$$-\frac{1}{2}\frac{1}{2\pi^2}\lim_{R\to\infty}\int\frac{2\sigma-1}{1+(\sigma-1)LG_{c,\frac{\omega}{Z}}}\text{p.v.}\int\frac{LG_{c,\frac{\omega'}{Z}}}{\omega'^2-\omega^2}d\omega'd\omega.$$

These two, of course, cancel, leading to the conclusion that

$$\lim_{R\to\infty}\frac{y_1^{(P)}(0)}{y_0^{(P)}}=0. \tag{94}$$

M.7 Conclusion

For $\delta<\frac{1}{2}$, the inequalities (91) and (85) together imply

$$\lim_{R\to\infty}\frac{y_1^{(P)}(0)}{y_1^{(P)}\left(\frac{\pi}{2}\right)}\geq\frac{\sigma-1}{2\sigma-1}(1-2\delta).$$

This is the result presented in (47). For $\delta>\frac{1}{2}$, (94) and (84) give

$$\lim_{R\to\infty}\frac{y_1^{(P)}(0)}{y_1^{(P)}\left(\frac{\pi}{2}\right)}=0.$$

This is the result (48).